CHAPTER 8

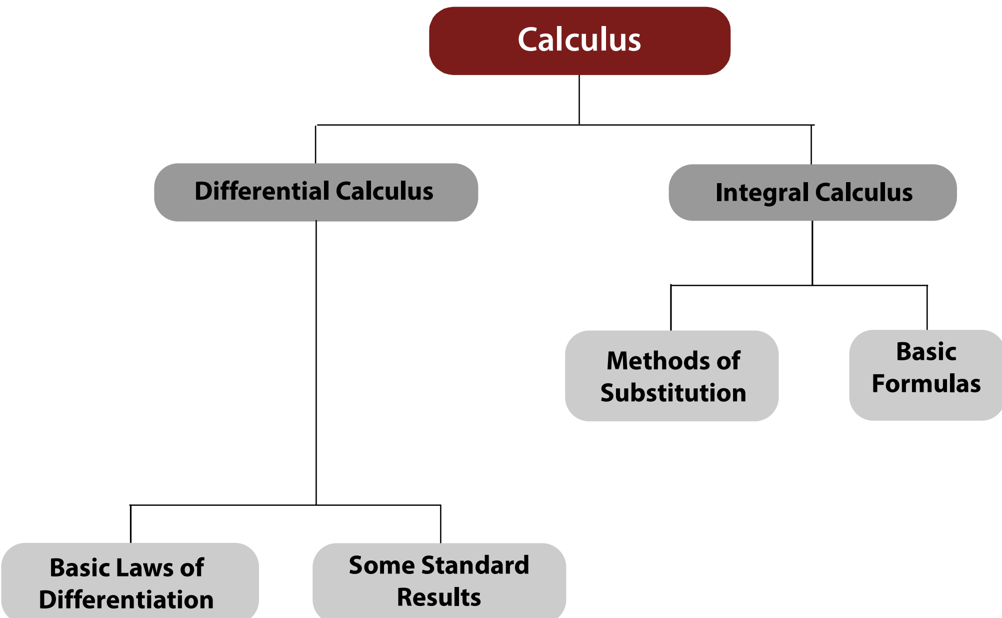
BASIC CONCEPTS OF DIFFERENTIAL AND INTEGRAL CALCULUS

LEARNING OBJECTIVES

After reading this chapter, students will be able to understand:

- Understand the basics of differentiation and integration.
- Know how to compute derivative of a function by the first principle, derivative of a function by the application of formulae and higher order differentiation.
- Appreciate various techniques of integration.
- Understand the concept of definite of integrals of functions and its application.

CHAPTER OVERVIEW



INTRODUCTION TO DIFFERENTIAL AND INTEGRAL CALCULUS (EXCLUDING TRIGONOMETRIC FUNCTIONS)

(A) DIFFERENTIAL CALCULUS

8.A.1 INTRODUCTION

Differentiation is one of the most important fundamental operations in calculus. Its theory primarily depends on the idea of limit and continuity of function.

To express the rate of change in any function we introduce concept of derivative which involves a very small change in the dependent variable with reference to a very small change in independent variable.

Thus differentiation is the process of finding the derivative of a continuous function. It is defined as the limiting value of the ratio of the change (increment) in the function corresponding to a small change (increment) in the independent variable (argument) as the later tends to zero.

8.A.2 DERIVATIVE OR DIFFERENTIAL COEFFICIENT

Let y = f(x) be a function. If h (or Δx) be the small increment in x and the corresponding increment in y or f(x) be $\Delta y = f(x+h) - f(x)$ then the derivative of f(x) is defined

as $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ i.e.

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

This is denoted as $f'(x)$ or dy/dx or $\frac{d}{dx}f(x)$. The derivative of f(x) is also known as differential coefficient of f(x) with respect to x. This process of differentiation is called the first principle (or definition or abinitio) (Ab-initio).

Note: In the light of above discussion a function f (x) is said to differentiable at x = c if

$\lim_{h \to c} \frac{f(x)-f(c)}{x-c}$ exist which is called the differential coefficient of f(x) at x = c and is denoted

by $f'(c)$ or $\left[\frac{dy}{dx}\right]_{x=c}$.

We will now study this with an example.

Consider the function $f(x) = x^2$.

By definition

$$\frac{d}{dx}f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x + 0 = 2x$$

Thus, derivative of f(x) exists for all values of x and equals 2x at any point x.

Examples of differentiations from the 1st principle

i) f(x) = c, c being a constant.

Since c is constant we may write f(x+h) = c.

So f(x+h) – f(x) = 0

Hence $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$

So $\frac{d(c)}{dx} = 0$

ii) Let $f(x) = x^n$; then $f(x+h) = (x+h)^n$

let x+h =t or h= t – x and as h→0, t→x

$$\text{Now } f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{t \to x} (t^n - x^n) / (t - x) = nx^{n-1}$$

Hence $\frac{d}{dx}(x^n) = nx^{n-1}$

iii) $f(x) = e^x \therefore f(x+h) = e^{x+h}$

$$\text{So } f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x(e^h - 1)}{h}$$

$$= e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x.1$$

Hence $\frac{d}{dx}(e^x) = e^x$

iv) Let $f(x) = a^x$ then $f(x+h) = a^{x+h}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{a^{x+h}-a^x}{h} = \lim_{h\to 0} \left[\frac{a^x(a^h-1)}{h}\right]$$

$$= a^x \lim_{h\to 0} \frac{a^h-1}{h}$$

$$= a^x \log_e a$$

Thus $\frac{d}{dx}(a^x) = a^x \log_e a$

v) Let $f(x) = \sqrt{x}$. Then $f(x+h) = \sqrt{x+h}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Thus $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

vi) $f(x) = \log x \therefore f(x+h) = \log(x+h)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\log(x+h)-\log x}{h}$$

$$= \lim_{h\to 0} \frac{\log\left(\frac{x+h}{x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left\{\log\left(1+\frac{h}{x}\right)\right\}$$

Let $\frac{h}{x} = t$ i.e. $h=tx$ and as $h \to 0,\ t \to 0$

$$\therefore f'(x) = \lim_{t\to 0} \frac{1}{tx}\log(1+t) = \frac{1}{x} \lim_{t\to 0} \frac{1}{t}\log(1+t) = \frac{1}{x}\times 1 = \frac{1}{x}, \text{ since } \lim_{t\to 0} \frac{\log(1+t)}{t} = 1$$

Thus $\frac{d}{dx}(\log x) = \frac{1}{x}$

8.A.3 SOME STANDARD RESULTS (FORMULAS)

(1) $\frac{d}{dx}(x^n) = nx^{n-1}$ (2) $\frac{d}{dx}(e^x) = e^x$ (3) $\frac{d}{dx}(a^x) = a^x \log_e a$

(4) $\frac{d}{dx}(\text{constant}) = 0$ (5) $\frac{d}{dx}(e^{ax}) = ae^{ax}$ (5) $\frac{d}{dx}(\log x) = \frac{1}{x}$

Note: $\frac{d}{dx}\{c\, f(x)\} = cf'(x)$ c being constant.

In brief we may write below the above functions and their derivatives:

Table: Few functions and their derivatives

Function	derivative of the function
$f(x)$	$f'(x)$
x^n	$n\,x^{n-1}$
e^{ax}	ae^{ax}
$\log x$	$1/x$
a^x	$a^x \log_e a$
c (a constant)	0

We also tabulate the basic laws of differentiation.

Table: Basic Laws for differentiation

Function	Derivative of the function
(i) h(x) = c.f(x) where c is any real constant (Scalar multiple of a function)	$\frac{d}{dx}\{h(x)\} = c.\frac{d}{dx}\{f(x)\}$
(ii) h(x) = f(x) ± g(x) (Sum/Difference of function)	$\frac{d}{dx}\{h(x)\} = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}\{g(x)\}$
(iii) h(x) = f(x). g(x) (Product of functions)	$\frac{d}{dx}\{h(x)\} = f(x)\frac{d}{dx}\{g(x)\} + g(x)\frac{d}{dx}\{f(x)\}$
(iv) $h(x) = \frac{f(x)}{g(x)}$ (Quotient of function)	$\frac{d}{dx}\{h(x)\} = \frac{g(x)\frac{d}{dx}\{f(x)\} - f(x)\frac{d}{dx}\{g(x)\}}{\{g(x)\}^2}$
(v) h(x) = f{g(x)}	$\frac{d}{dx}\{h(x)\} = \frac{d}{dz}f(z).\frac{dz}{dx}$, where z = g(x)

It should be noted here even though in (ii), (iii), (iv) and (v) we have considered two functions f and g, it can be extended to more than two functions by taking two by two.

Example: Differentiate each of the following functions with respect to x:

(a) $3x^2 + 5x - 2$ (b) $a^x + x^a + a^a$ (c) $\frac{1}{3}x^3 - 5x^2 + 6x - 2\log x + 3$

(d) $e^x \log x$ (e) $2^x x^5$ (f) $\frac{x^2}{e^x}$

(g) $e^x / \log x$ (h) $2^x . \log x$ (i) $\frac{2x}{3x^3+7}$

Solution: (a) Let $y = f(x) = 3x^2 + 5x - 2$

$$\frac{dy}{dx} = 3\frac{d}{dx}(x)^2 + 5\frac{d}{dx}(x) - \frac{d}{dx}(2)$$

$= 3 \times 2x + 5.1 - 0 = 6x + 5$

(b) Let $h(x) = a^x + x^a + a^a$

$$\frac{d}{dx}\{h(x)\} = \frac{d}{dx}(a^x + x^a + a^a) = \frac{d}{dx}(a^x) + \frac{d}{dx}(x^a) + \frac{d}{dx}(a^a)$$, a^a is a constant

$= a^x \log a + ax^{a-1} + 0 = a^x \log a + ax^{a-1}.$

(c) Let $f(x) = \frac{1}{3}x^3\text{-}5x^2\text{+}6x\text{-}2\log x\text{+}3 \quad \therefore \quad \frac{d}{dx}\{f(x)\}=\frac{d}{dx}\left(\frac{1}{3}x^3\text{-}5x^2\text{+}6x\text{-}2\log x\text{+}3\right)$

$= \frac{1}{3}.3x^2\text{-}5.2x\text{+}6.1\text{-}2.\frac{1}{x}\text{+}0 \quad = x^2\text{-}10x\text{+}6\text{-}\frac{2}{x}.$

(d) Let $y = e^x \log x$

$\frac{dy}{dx} = e^x \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(e^x)$ (Product rule)

$= \frac{e^x}{x} + e^x \log x = \frac{e^x}{x}(1+ x \log x)$

So $\frac{dy}{dx} = \frac{e^x}{x}(1 + x \log x)$

(e) $y = 2^x x^5$

$\frac{dy}{dx} = x^5 \frac{d}{dx}(2^x) + 2^x \frac{d}{dx}(x^5)$ (Product Rule)

$= x^5 2^x \log_e 2 + 2^x. 5. x^4$

(f) let $y = \frac{x^2}{e^x}$

$\frac{dy}{dx} = \frac{e^x \frac{8}{dx}(x^2) - x^2 \frac{d}{dx}(e^x)}{(e^x)^2}$ (Quotient Rule)

$= \frac{2xe^x \text{-} x^2 e^x}{(e^x)^2} = \frac{x(2\text{-}x)}{e^x}$

(g) Let $y = e^x / \log x$

so $\frac{dy}{dx} = \frac{(\log x)\frac{d}{dx}(e^x) - e^x \frac{d}{dx}(\log x)}{(\log x)^2}$ (Quotient Rule)

$= \frac{e^x \log x \text{-} e^x/x}{(\log x)^2} = \frac{e^x \; x \log x \text{-} e^x}{x(\log x)^2}$

So $\frac{dy}{dx} = \frac{e^x(x \log x \text{-} 1)}{x(\log x)^2}$

(h) Let $h(x) = 2^x. \log x$

The given function h(x) is appearing here as product of two functions

$f(x) = 2^x$ and $g(x) = \log x$.

$$= \frac{d}{dx}\{h(x)\} = \frac{d}{dx}(2^x.\log x) = 2^x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(2^x).$$

$$2^x \times \frac{1}{x} + \log x.(2^x \log 2) = \frac{2^x}{x} + 2^x \log 2 \log x$$

(i) Let $h(x) = \frac{2x}{3x^3+7}$ [Given function appears as the quotient of two functions]

$f(x) = 2x$ and $g(x) = 3x^3 + 7$

$$\frac{d}{dx}\{h(x)\} = \frac{(3x^3+7)\frac{d}{dx}(2x) - 2x\frac{d}{dx}(3x^3+7)}{(3x^3+7)^2} = \frac{(3x^3+7).2 - 2x.(9x^2+0)}{(3x^3+7)^2}$$

$$= \frac{2\left\{(3x^3+7) - 9x^3\right\}}{(3x^3+7)^2} = \frac{2(7-6x^3)}{(3x^3+7)^2}.$$

8.A.4 DERIVATIVE OF A FUNCTION OF FUNCTION

If $y = f[h(x)]$ then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = f'(u) \times h'(x)$ where $u = h(x)$

Example: Differentiate $\log(1 + x^2)$ wrt. x

Solution: Let $y = \log(1 + x^2) = \log t$ when $t = 1 + x^2$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{t} \times (0+2x) = \frac{2x}{t} = \frac{2x}{(1+x^2)}$$

This is an example of derivative of function of a function and the rule is called Chain Rule.

8.A.5 IMPLICIT FUNCTIONS

A function in the form $f(x, y) = 0$ is known as implicit function. For example $x^2y^2 + 3xy + y = 0$ where y cannot be directly defined as a function of x is called an implicit function of x.

In case of implicit functions if y be a differentiable function of x, no attempt is required to express y as an explicit function of x for finding out $\frac{dy}{dx}$. In such case differentiation of both sides with respect of x and substitution of $\frac{dy}{dx} = y_1$ gives the result. Thereafter y_1 may be obtained by solving the resulting equation.

Example: Find $\frac{dy}{dx}$ for $x^2y^2 + 3xy + y = 0$

Solution: $x^2y^2 + 3xy + y = 0$

Differentiating with respect to x we see

$$x^2 \frac{d}{dx}(y^2) + y^2 \frac{d}{dx}(x^2) + 3x \frac{d(y)}{dx} y + 3y \frac{d}{dx}(x) + \frac{dy}{dx} = 0$$

$$\text{or } 2yx^2 \frac{dy}{dx} + 2xy^2 + 3x \frac{dy}{dx} + 3y \frac{d(x)}{dx} + \frac{dy}{dx} = 0, \quad \frac{d}{dx}(x) = 1, \frac{d(y^2)}{dx} = 2y\frac{dy}{dx} \text{ (chain rule)}$$

$$\text{or } (2yx^2 + 3x + 1)\frac{dy}{dx} + 2xy^2 + 3y = 0$$

$$\text{or } \frac{dy}{dx} = -\frac{(2xy^2 + 3y)}{(2x^2y + 3x + 1)}$$

This is the procedure for differentiation of Implicit Function.

8.A.6 PARAMETRIC EQUATION

When both the variables x and y are expressed in terms of a parameter (a third variable), the involved equations are called parametric equations.

For the parametric equations $x = f(t)$ and $y = h(t)$ the differential coefficient $\frac{dy}{dx}$ is obtained by using $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}. = \frac{dy}{dt} \cdot \frac{dt}{dx}$

Example: Find $\frac{dy}{dx}$ if $x = at^3, y = a / t^3$

Solution: $\frac{dx}{dt} = 3at^2$; $\quad \frac{dy}{dt} = -3a / t^4$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{-3a}{t^4} \times \frac{1}{3at^2} = \frac{-1}{t^6}$$

This is the procedure for differentiation of parametric functions.

8.A.7 LOGARITHMIC DIFFERENTIATION

The process of finding out derivative by taking logarithm in the first instance is called logarithmic differentiation. The procedure is convenient to adopt when the function to be differentiated involves a function in its power or when the function is the product of number of functions.

Example: Differentiate x^x w.r.t. x

Solution: let $y = x^x$

Taking logarithm,

log y = x log x

Differentiating with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = \log\ x + \frac{x}{x} = 1 + \log x$$

or $\frac{dy}{dx} = y\,(1 + \log x) = x^x\,(1 + \log x)$

This procedure is called logarithmic differentiation.

8.A.8 SOME MORE EXAMPLES

(1) If $y = \sqrt{\frac{1-x}{1+x}}$ show that $(1 - x^2)\frac{dy}{dx} + y = 0.$

Solution: Taking logarithm, we may write $\log y = \frac{1}{2}\{\log(1 - x) - \log(1 + x)\}$

Differentiating throughout we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}\{\log(1 - x) - \log(1 + x)\} = \frac{1}{2}\left(\frac{-1}{1-x} - \frac{1}{1+x}\right) = -\frac{1}{1-x^2}.$$

By cross–multiplication $(1 - x^2)\frac{dy}{dx} = -y$

Transposing $(1 - x^2)\frac{dy}{dx} + y = 0.$

(2) Differentiate the following w.r.t. x:

(a) $\log(x + \sqrt{x^2 + a^2})$

(b) $\log\left(\sqrt{x-a} + \sqrt{x-b}\right).$

Solution: (a) $y = \log(x + \sqrt{x^2 + a^2})$

$$\frac{dy}{dx} = \frac{1}{(x+\sqrt{x^2+a^2})}\left(1 + \frac{1}{2\sqrt{x^2+a^2}}(2x)\right)$$

$$= \frac{1}{(x+\sqrt{x^2+a^2})} + \frac{x}{(x+\sqrt{x^2+a^2})\sqrt{x^2+a^2}}$$

$$= \frac{(x+\sqrt{x^2+a^2})}{(x+\sqrt{x^2+a^2})\ \sqrt{x^2+a^2}} = \frac{1}{\sqrt{x^2+a^2}}$$

(b) Let $y = \log(\sqrt{x-a}+\sqrt{x-b})$

or $\frac{dy}{dx} = \frac{1}{\sqrt{x-a}+\sqrt{x-b}}\left(\frac{1}{2\sqrt{x-a}}+\frac{1}{2\sqrt{x-b}}\right) = \frac{(\sqrt{x-b}+\sqrt{x-a})}{(\sqrt{x-a}+\sqrt{x-b})\ 2\sqrt{x-a}\sqrt{x-b}}$.

$= \frac{1}{2\sqrt{x-a}\sqrt{x-b}}$

(3) If $x^m y^n = (x+y)^{m+n}$ prove that $\frac{dy}{dx} = \frac{y}{x}$

Solution: $x^m y^n = (x+y)^{m+n}$

Taking log on both sides

$\log x^m y^n = (m+n) \log (x + y)$

or $m \log x + n \log y = (m+n) \log (x+y)$

so $\frac{m}{x}+\frac{n}{y}\frac{dy}{dx} = \frac{(m+n)}{(x+y)}\left(1+\frac{dy}{dx}\right)$

or $\left(\frac{n}{y}-\frac{m+n}{x+y}\right)\frac{dy}{dx} = \frac{m+n}{(x+y)}-\frac{m}{x}$

or $\frac{(nx+ny-my-ny)}{y(x+y)}\ \frac{dy}{dx} = \frac{mx+nx-mx-my}{x(x+y)}$

or $\frac{(nx-my)}{y}\ \frac{dy}{dx} = \frac{nx-my}{x}$

or $\frac{dy}{dx} = \frac{y}{x}$ proved.

(4) If $x^y = e^{x-y}$ prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

Solution: $x^y = e^{x-y}$

So $y \log x = (x - y) \log e$

or $y \log x = (x - y)$ ....(a)

Differentiating w.r.t. x we get

$$\frac{y}{x} + \log x \frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\text{or } (1 + \log x)\frac{dy}{dx} = 1 - \frac{y}{x}$$

or $\frac{dy}{dx} = \frac{(x-y)}{x(1+\log x)}$, substituting x–y = log x, from (a) we have

$$\text{or } \frac{dy}{dx} = \frac{y(\log x)}{x(1+\log x)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(b)}$$

From (a) $y(1 + \log x) = x$

$$\text{or } \frac{y}{x} = \frac{1}{(1+\log x)}$$

$$\text{From (b) } \frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$$

8.A.9 BASIC IDEA ABOUT HIGHER ORDER DIFFERENTIATION

Let $y = f(x) = x^4 + 5x^3 + 2x^2 + 9$

$$\frac{dy}{dx} = \frac{d}{dx}f(x) = 4x^3 + 15x^2 + 4x = f'(x)$$

Since f(x) is a function of x it can be differentiated again.

$$\text{Thus } \frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x) = \frac{d}{dx}(4x^3 + 15x^2 + 4x) = 12x^2 + 30x + 4$$

$\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is written as $\frac{d^2y}{dx^2}$ (read as d square y by dx square) and is called the second derivative of y with respect to x while $\frac{dy}{dx}$ is called the first derivative. Again the second derivative here being a function of x can be differentiated again and $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$

$$= f'''(x) = 24x + 30.$$

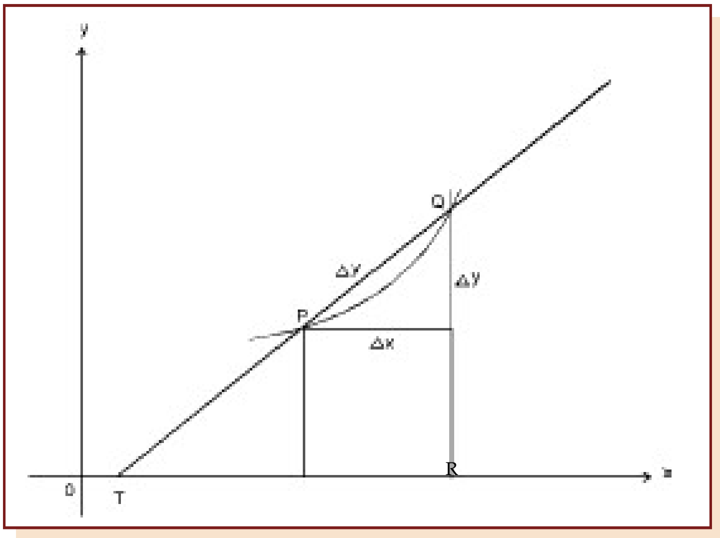
Example: If $y = ae^{mx} + be^{-mx}$ prove that $\frac{d^2y}{dx^2} = m^2y$.

Solution: $\frac{dy}{dx} = \frac{d}{dx}(ae^{mx} + be^{-mx}) = ame^{mx} - bme^{-mx}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(ame^{mx} - bme^{-mx})$$

$$= am^2e^{mx} + bm^2e^{-mx} = m^2(ae^{mx} + be^{-mx}) = m^2y.$$

8.A.10 GEOMETRIC INTERPRETATION OF THE DERIVATIVE



Let f(x) represent the curve in the fig. We take two adjacent pairs P and Q on the curve Let f(x) represent the curve in the fig. We take two adjacent points P and Q on the curve whose coordinates are (x, y) and $(x + \Delta x, y+\Delta y)$ respectively. The slope of the chord TPQ is given by

$\Delta y/\Delta x$ when $\Delta x \to 0$, $Q \to P$. TPQ becomes the tangent at P and $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

The derivative of f(x) at a point x represents the slope (or sometime called the gradient of the curve) of the tangent to the curve y = f(x) at the point x. If $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ exists for a particular point say x =a and f(a) is finite we say the function is differentiable at x = a and continuous at that point.

Example: Find the gradient of the curve $y = 3x^2 - 5x + 4$ at the point (1, 2).

Solution: $y = 3x^2 - 5x + 4 \quad \therefore \frac{dy}{dx} = 6x - 5$

so $[dy/dx]_{x=1\ y=2} = 6.1 - 5 = 6 - 5 = 1$

Thus the gradient of the curve at (1, 2) is 1.

SUMMARY

- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(a^x) = a^x \log_e a$

- $\frac{d}{dx}$ (constant) = 0
- $\frac{d}{dx}(e^{ax}) = ae^{ax}$
- $\frac{d}{dx}(\log x) = \frac{1}{x}$

Note: $\frac{d}{dx}\{c\,f(x)\} = cf'(x)$ c being constant.

Function	Derivative of the function
(i) h(x) = c.f(x) where c is any real constant (Scalar multiple of a function)	$\frac{d}{dx}\{h(x)\} = c.\frac{d}{dx}\{f(x)\}$
(ii) h(x) = f(x) ± g(x) (Sum/Difference of function)	$\frac{d}{dx}\{h(x)\} = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}\{g(x)\}$
(iii) h(x) = f(x). g(x) (Product of functions)	$\frac{d}{dx}\{h(x)\} = f(x)\frac{d}{dx}\{g(x)\} + g(x)\frac{d}{dx}\{f(x)\}$
(iv) $h(x) = \frac{f(x)}{g(x)}$ (Quotient of function)	$\frac{d}{dx}\{h(x)\} = \frac{g(x)\frac{d}{dx}\{f(x)\} - f(x)\frac{d}{dx}\{g(x)\}}{\{g(x)\}^2}$
(v) h(x) = f{g(x)}	$\frac{d}{dx}\{h(x)\} = \frac{d}{dz}f(z).\frac{dz}{dx}$, where z = g(x)

Applications of Differential Calculus:

In this chapter we have discussed the concept of differentiation. Differentiation helps us to find out the average rate of change in the dependent variable with respect to change in the independent variable. It makes differentiation to have applications. Various scientific formulae and results involves rate of change in price, change in demand with respect change in output, change in revenue obtained with respect to change in price, change in demand with respect change in income, etc.

1) Ate of Change in Quantities: Let there two variables x and y such that y is a function of x.

   Differential coefficient $\frac{dy}{dx}$ represents the rate of change of y with respect to x.

2) In other words, the expression, "the rate of Change of a function" means the derivative of the function.

3) We write f(x) in place of y and f′(x) in place of $\frac{dy}{dx}$.

Cost Function: Total cost consists of two parts (i) Variable Cost (ii) Fixed Cost.

If C(X) denotes the cost producing x units of a product then C(x) = V(x) + F(x) , where V(x) denotes the variable cost and F(x) is the fixed cost. Variable cost depends upon the number of units produced (i.e value of x) whereas fixed cost is independent of the level of output x. For example,

Average cost (AC or $\bar{C}$) = $\frac{\text{Total Cost}}{\text{OutPut}} = \frac{C(X)}{X}$

Average variable cost (AVC) = $\frac{\text{Variable Cost}}{\text{OutPut}} = \frac{V(x)}{x}$

Average Fixed Cost (AFC) = $\frac{\text{Fixed Cost}}{\text{OutPut}} = \frac{F(x)}{x}$

Marginal Cost: If C(x) the total cost producing x units then the increase in cost in producing one more unit is called marginal cost at an output level of x units and is given as $\frac{dC}{dx}$.

Marginal Cost (MC) = Rate of change in cost C per unit change in OutPut at an OutPut level of x units = $\frac{dC}{dx}$.

To increase profits of a company may decide to increase its production. The question that concerns the management is how will the cost be affected by an increase in production. Economists use the marginal cost to answer the question.

Example 1: The total cost function of a firm is $C(x) = \frac{1}{3}x^3 - 5x^2 + 28x + 10$ where C is the total cost and x is outpout.

A tax at the rate of ₹ 2 per unit of output is imposed and the producer adds it to his cost. If the market demand function is given by, where ₹ p is the price per unit of output, find the profit maximising output and price for maximum profit.

Solution:

After the imposition of tax of ₹ 2 per unit, the total new cost is

$$C(x) = \frac{1}{3}x^3 - 5x^2 + 28x + 10 + 2x$$

Also, $R(x) = px = (2530 - 5x)x = 2530x - 5x^2$

$\therefore\ P(x) = R(x) - C(x)$

$$= (2530x - 5x^2) - (\frac{1}{3}x^3 - 5x^2 + 30x + 10) \;=\; -\frac{1}{3}x^3 + 2500x - 10$$

For maximum total profit, and .

P′(x) = 0 gives $-x^2 + 2500 = 0 \ \therefore\ x = \pm 50$

Since output cannot be negative, we consider x = 50.

For x = 50, $P''(x) = -2x = -2 \times 50 = -100 < 0$

Thus, the profit is maximum at x = 50.

Putting x = 50 in the demand function, the corresponding price is $p = 2530 - 5 \times 50 =$ ₹ 2280.

Example 2: The cost function of a company is given by:

$$C(x) = 100x - 8x^2 + \frac{x^3}{3},$$

where x denotes the output. Find the level of output at which:

(i) marginal cost is minimum

(ii) average cost is minimum

Solution:

$$M(x) = \text{Marginal Cost} = C(x) = \frac{d}{dx}\left(100x - 8x^2 + \frac{x^3}{3}\right) = 100 - 16x + x^2$$

$$A(x) = \text{Average Cost} = \frac{C(x)}{x} = 100 - 8x + \frac{x^2}{3}.$$

(i) M(x) is maximum or minimum when M¢(x) = -16 + 2x = 0 or, x = 8.

$M²(8) = M''(x)]_{x=8} = [2]_{x=8} = 2 > 0$

Hence, marginal cost is minimum at x = 8.

(ii) A(x) is maximum or minimum when $A'(x) = -8 + \frac{2x}{3} = 0$ or, x = 12.

$A''(12)\ A''(x)]_{x=12} = \left.\frac{2}{3}\right]_{x=12} = \frac{2}{3} > 0$

Hence, average cost is minimum at x = 12.

$$A(x) = \text{Average Cost} = 100 - 8x + \frac{x^2}{3} = 100 - 8(12) + \frac{144}{3}$$

$$= 100 - 96 + 48 = 52$$

1) **Revenue Function**: Revenue, R(x), gives the total money obtained (Total turnover) by selling x units of a product. If x units are sold at 'P per unit, then R(x) = P.X

Marginal Revenue: It is the rate of change I revenue per unit change in output. If R is the revenue and x is the output, then $MR = \frac{dR}{dx}$.

Profit function: Profit P(x), the difference of between total revenue R(x) and total Cost C (x).

P(X)= R(x) – C(x)

Marginal Profit: It is rate of change in profit per unit change in output. i.e $\frac{dP}{dx}$

Example 3: A computer software company wishes to start the production of floppy disks. It was observed that the company had to spend ₹ 2 lakhs for the technical informations. The cost of setting up the machine is ₹ 88,000 and the cost of producing each unit is ₹ 30, while each floppy could be sold at ₹ 45. Find:

(i) the total cost function for producing x floppies; and

(ii) the break-even point.

Solution :

a) Given, fixed cost = ₹ 2,00,000 + ₹ 88,000 = ₹ 2,88,000.

(i) If C (x) be the total cost function for producing floppies, then C(x) = 30x + 2,88,000

(ii) The Revenue function R(x), for sales of x floppies is given by R(x) = 45x.

For break-even point, R(x) = C(x)

i.e., 45x = 30x + 2,88,000

i.e., 15x = 2,88,0000 $\therefore$ x = 19,200, the break-even point

Example 4: A company decided to set up a small production plant for manufacturing electronic clocks. The total cost for initial set up (fixed cost) is ₹ 9 lakhs. The additional cost for producing each clock is ₹ 300. Each clock is sold at ₹ 750. During the first month, 1,500 clocks are produced and sold.

(i) What profit or loss the company incurs during the first month, when all the 1,500 clocks are sold ?

(ii) Determine the break-even point.

(b) Total cost of producing 20 items of a commodity is ₹ 205, while total cost of producing 10 items is ₹ 135. Assuming that the cost function is a linear function, find the cost function and marginal cost function.

Solution:

(a) The total cost function for manufacturing x Clocks is given by C(x) = Fixed cost + Variable cost to produce x Clocks = 9,00,000 + 300x.

The revenue function from the sale of x clocks in given by R(x) = 750 × x = 750x.

(i) Profit function,

P(x) = R(x) – C(x)

= 750x – (9,00,000 + 300x) = 450x – 9,00,000

∴ Profit, when all 1500 clocks are sold = P(1500) = 450 × 1500 – 9,00,000 = – ₹ 2,25,000

Thus, there is a loss of ‘ 2,25,000 when only 1500 clocks are sold.

(ii) At the break-even point, R(x) = C(x)

or, 9,00,000 + 300x = 750x

or, 450x = 9,00,000 ∴ x = 2,000

Hence, 2000 clocks have to be sold to achieve the break-even point.

(b) Let cost function be

C(x) = ax + b,…………(i)

x being number of items and a, b being constants.

Given, C(x) = 205 for x = 20 and C(x) = 135 for x = 10.

Putting these values in (i),

205 = 20a+b …………(ii)

135 = 10a+b …………(iii)

(ii) – (iii) gives,

70 = 10a or, a = 7

From (iii), b = 135 – 10a = 135 – 70 = 65

∴ Required cost function is given by C(x) = 7x + 65

Marginal cost function, $C'(x) = \frac{d}{dx}(7x+65) = 7$

Marginal Propensity to Consume (MPC): The consumption function C = F(Y) expresses the relationship between the total consumption and total Income (Y), then the marginal propensity to consume is defined as the rate of Change consumption per unit change in Income i.e., $\frac{dC}{dY}$. By consumption we mean expenditure incurred in on Consumption.

Marginal Propensity to save (MPS): Saving, S is the difference between income, I and consumption, c, i.e ., $\frac{dS}{dY}$.

EXERCISE 8(A)

Choose the most appropriate option (a) (b) (c) or (d).

1. The gradient of the curve $y = 2x^3 - 3x^2 - 12x + 8$ at $x = 0$ is

   a) –12 b) 12 c) 0 d) none of these

2. The gradient of the curve $y = 2x^3 - 5x^2 - 3x$ at $x = 0$ is

   a) 3 b) –3 c) 1/3 d) none of these

3. The derivative of $y = \sqrt{x+1}$ is

   a) $1/\sqrt{x+1}$ b) $-1/\sqrt{x+1}$ c) $1/2\sqrt{x+1}$ d) none of these

4. If $f(x) = e^{ax^2+bx+c}$ the $f'(x)$ is

   a) e^{ax^2+bx+c} b) $e^{ax^2+bx+c}(2ax+b)$ c) $2ax+b$ d) none of these

5. If $f(x) = \frac{x^2+1}{x^2-1}$ then $f'(x)$ is

   a) $-4x/(x^2-1)^2$ b) $4x/(x^2-1)^2$ c) $x/(x^2-1)^2$ d) none of these

6. If $y = x(x-1)(x-2)$ then $\frac{dy}{dx}$ is

   a) $3x^2 - 6x + 2$ b) $-6x + 2$ c) $3x^2 + 2$ d) none of these

7. If $xy = 1$ then $y^2 + dy/dx$ is equal to

   a) 1 b) 0 c) –1 d) none of these

8. The derivative of the function $\sqrt{x+\sqrt{x}}$ is

   a) $\frac{1}{2\sqrt{x+\sqrt{x}}}$ b) $1+\frac{1}{2\sqrt{x}}$ c) $\frac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\frac{1}{2\sqrt{x}}\right)$ d) none of these

9. Given $e^{-xy} - 4xy = 0$, $\frac{dy}{dx}$ can be proved to be

   a) $-y/x$ b) y/x c) x/y d) none of these

10. If $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$, $\frac{dy}{dx}$ can be expressed as

    a) $\frac{x}{y}$ b) $\frac{x}{\sqrt{x^2-a^2}}$ c) $\frac{1}{\sqrt{\frac{x^2}{a^2}-1}}$ d) none of these

11. If $\log (x / y) = x + y$, $\frac{dy}{dx}$ may be found to be

a) $\frac{y(1-x)}{x(1+y)}$ b) $\frac{y}{x}$ c) $\frac{1-x}{1+y}$ d) none of these

12. If $f(x, y) = x^3 + y^3 - 3axy = 0$, $\frac{dy}{dx}$ can be found out as

a) $\frac{ay-x^2}{y^2+ax}$ b) $\frac{ay-x^2}{y^2-ax}$ c) $\frac{ay+x^2}{y^2+ax}$ d) none of these

13. Given $x = at^2$, $y = 2at$; $\frac{dy}{dx}$ is calculated as

a) t b) $-1/t$ c) $1/t$ d) none of these

14. Given $x = 2t + 5$, $y = t^2 - 2$; $\frac{dy}{dx}$ is calculated as

a) t b) $-1/t$ c) $1/t$ d) none of these

15. If $y = \frac{1}{\sqrt{x}}$ then $\frac{dy}{dx}$ is equal to

a) $\frac{1}{2x\sqrt{x}}$ b) $\frac{-1}{x\sqrt{x}}$ c) $-\frac{1}{2x\sqrt{x}}$ d) none of these

16. If $x = 3t^2 - 1$, $y = t^3 - t$, then $\frac{dy}{dx}$ is equal to

a) $\frac{3t^2-1}{6t}$ b) $3t^2-1$ c) $\frac{3t-1}{6t}$ d) none of these

17. For the curve $x^2 + y^2 + 2gx + 2hy = 0$, the value of $\frac{dy}{dx}$ at (0, 0) is

a) $-g/h$ b) g/h c) h/g d) none of these

18. If $y = \frac{e^{3x}-e^{2x}}{e^{3x}+e^{2x}}$, then $\frac{dy}{dx}$ is equal to

a) $2e^{5x}$ b) $1/(e^{5x} + e^{2x})^2$ c) $e^{5x}/(e^{5x} + e^{2x})$ d) none of these

19. Given $x = t + t^{-1}$ and $y = t - t^{-1}$ the value of $\frac{dy}{dx}$ at $t = 2$ is

a) $3/5$ b) $-3/5$ c) $5/3$ d) none of these

20. If $x^3 - 2x^2 y^2 + 5x + y - 5 = 0$ then $\frac{dy}{dx}$ at x = 1, y = 1 is equal to

a) 4/3 b) – 4/3 c) 3/4 d) none of these

21. The derivative of $x^2 \log x$ is

a) 1+2log x b) x(1 + 2 log x) c) 2 log x d) none of these

22. The derivative of $\frac{3-5x}{3+5x}$ is

a) $30/(3+5x)^2$ b) $1/(3+5x)^2$ c) $-30/(3+5x)^2$ d) none of these

23. Let $y = \sqrt{2x} + 3^{2x}$ then $\frac{dy}{dx}$ is equal to

a) $(1/\sqrt{2x}) + 2.3^{2x}\log_e 3$ b) $1/\sqrt{2x}$

c) $2.3^{2x}\log_e 3$ d) none of these

24. The derivative of e^{3x^2-6x+2} is

a) $30(1-5x)^5$ b) $(1-5x)^5$ c) $6(x-1)e^{3x^2-6x+2}$ d) none of these

25. If $y = \frac{e^x+1}{e^x-1}$ then $\frac{dy}{dx}$ is equal to

a) $\frac{-2e^x}{(e^x-1)^2}$ b) $\frac{2e^x}{(e^x-1)^2}$ c) $\frac{-2}{(e^x-1)^2}$ d) none of these

26. If $x = at^2$, $y = 2at$ then $\left[\frac{dy}{dx}\right]_{t=2}$ is equal to

a) 1/2 b) –2 c) –1/2 d) none of these

27. Let $f(x) = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$ then f′(2) is equal to

a) 3/4 b) 1/2 c) 0 d) none of these

28. If $f(x) = x^2 - 6x + 8$ then f′(5) – f′(8) is equal to

a) f′(2) b) 3f′(2) c) 2f′(2) d) none of these

29. If $f(x) = x^k$ and f′(1) = 10 the value of k is

a) 10 b) –10 c) 1/10 d) none of these

30. If $y = e^x + e^{-x}$ then $\frac{dy}{dx} - \sqrt{y^2 - 4}$ is equal to

a) 1 b) –1 c) 0 d) none of these

31. The derivative of $(x^2-1)/x$ is

a) $1 + 1/x^2$ b) $1 - 1/x^2$ c) $1/x^2$ d) none of these

32. The differential coefficients of $(x^2 + 1)/x$ is

a) $1 + 1/x^2$ b) $1 - 1/x^2$ c) $1/x^2$ d) none of these

33. If $y = e^{\sqrt{2x}}$ then $\frac{dy}{dx}$ is equal to _____________.

a) $\frac{e^{\sqrt{2x}}}{\sqrt{2x}}$ b) $e^{\sqrt{2x}}$ c) $\frac{e^{\sqrt{2x}}}{\sqrt{2x}}$ d) none of these

34. If $x = (1 - t^2)/(1 + t^2)$ $y = 2t/(1 + t^2)$ then dy/dx at $t = 1$ is _____________.

a) 1/2 b) 1 c) 0 d) none of these

35. $f(x) = x^2/e^x$ then $f'(1)$ is equal to _____________.

a) $-1/e$ b) $1/e$ c) e d) none of these